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Wigner's D-matrix elements for $SU(3)$—a generating function approach

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Abstract. Using our calculus for $SU(3)$ a generating function for Wigner D-matrix elements is derived for the first time. From this an explicit expression for the individual matrix elements, in an arbitrary irreducible representation, is obtained, also for the first time, in terms of polynomials in the matrix elements of the defining representation of $SU(3)$. This expression does not depend on any particular parametrization of the group.

1. Introduction

The Wigner D-matrix elements of $SU(3)$ have very important applications in nuclear physics, particle physics, $SU(3)$ lattice gauge theories, matrix models, finite temperature field theory calculations involving $SU(3)$ and other areas of physics. Starting with Murnaghan [2], who parametrized the defining matrices of $U(n)$ and $O(n)$, many authors [4–7] have obtained expressions for the Wigner D-matrix elements of $SU(3)$ using various methods. It is the purpose of this paper to evaluate these matrix elements for $SU(3)$ using the calculus [1, 11] we have set up to deal with computations involving the group $SU(3)$. The distinct advantage of this calculus and the novelty of our present method is that it allows one to write a generating function for these matrix elements from which one can extract the individual matrix elements by using the auxiliary inner product of the calculus.

The layout of the paper is as follows. We begin, in section 2, by reviewing the main ingredients of our calculus for $SU(3)$ which are relevant to our present discussion and then, in section 3, give a derivation of the generating function for the matrix elements. In section 4 we show how to extract the individual matrix elements and obtain a polynomial expression for the matrix elements in any irreducible representation in terms of the matrix elements of the defining representation of $SU(3)$ in any parametrization. Section 5 is devoted to a discussion of our results. A few examples are included in the appendix for illustrating the method.

2. Overview of our previous results

In this section we briefly review the results that we need for the group $SU(3)$. Some of these results were obtained by us in a previous paper [1].

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$SU(3)$ is the group of 3×3 unitary unimodular matrices A with complex coefficients. It is a group of eight real parameters. The matrix elements satisfy the following conditions:

$$\begin{aligned} &A = (a_{ij}) \\ &A^\dagger A = I \qquad AA^\dagger = I \quad \text{where } I \text{ is the identity matrix and} \\ &\det(A) = 1\,. \end{aligned} \tag{1}$$

2.1. Parametrization

One well known parametrization of $SU(3)$ is due to Murnaghan [2], see also [3–5, 8]. In this we write a typical element of $SU(3)$ as

$$D(\delta_1, \delta_2, \phi_3)U_{23}(\phi_2, \sigma_3)U_{12}(\theta_1, \sigma_2)U_{13}(\phi_1, \sigma_1) \tag{2}$$

with the condition $\phi_3 = -(\delta_1 + \delta_2)$. Here D is a diagonal matrix whose elements are $\exp(\mathrm{i}\delta_1)$, $\exp(\mathrm{i}\delta_2)$, $\exp(\mathrm{i}\phi_3)$ and $U_{pq}(\phi, \sigma)$ is a 3×3 unitary unimodular matrix which, for instance, in the case $p = 1$, $q = 2$ has the form

$$\begin{pmatrix} \cos\phi & -\sin\phi\,\exp(-\mathrm{i}\sigma) & 0 \\ \sin\phi\,\exp(\mathrm{i}\sigma) & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{3}$$

The three parameters ϕ_1, ϕ_2, ϕ_3 are longitudinal angles whose range is $-\pi \leqslant \phi_i \leqslant \pi$, and the remaining six parameters are latitude angles whose range is $\frac{1}{2}\pi \leqslant \sigma_i \leqslant \frac{1}{2}\pi$.

Now the transformations U_{23} and U_{13} can be changed into transformations of the type U_{12} whose matrix elements are known, by the following device:

$$\begin{aligned} U_{13}(\phi_1, \sigma_1) &= (2,3)U_{12}(\phi_1, \sigma_1)(2,3) \\ U_{23}(\phi_2, \sigma_3) &= (1,2)(2,3)U_{12}(\phi_2, \sigma_3)(2,3)(1,2) \end{aligned} \tag{4}$$

where $(1, 2)$ and $(2, 3)$ are the transposition matrices

$$(1,2) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad (2,3) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{5}$$

In this way the expression for an element of the $SU(3)$ group becomes

$$D(\delta_1, \delta_2, \phi_3)(1,2)(2,3)U_{12}(\phi_2, \sigma_3)(2,3)(1,2)U_{12}(\theta_1, \sigma_2)(2,3)U_{12}(\phi_1, \sigma_1)(2,3)\,. \tag{6}$$

2.2. Irreducible representations

The above parametrization provides us with a defining irreducible representation $\underline{3}$ of $SU(3)$ acting on a three-dimensional complex vector space spanned by the triplet z_1, z_2, z_3 of complex variables. The Hermitian adjoint of the above matrix gives us another defining but inequivalent irreducible representation $\underline{3^*}$ of $SU(3)$ acting on the triplet w_1, w_2, w_3 of complex variables spanning another three-dimensional complex vector space. Tensors constructed out of these two three-dimensional representations span an infinite-dimensional complex vector space.

2.3. The constraint

If we impose the constraint

$$z_1 w_1 + z_2 w_2 + z_3 w_3 = 0 \tag{7}$$

on this space we obtain an infinite-dimensional complex vector space in which each irreducible representation of $SU(3)$ occurs once and only once. Such a space is called a model space for $SU(3)$. Furthermore, if we solve the constraint $z_1w_1+z_2w_2+z_3w_3 = 0$ and eliminate one of the variables, say w_3, in terms of the other five variables z_1, z_2, z_3, w_1, w_2 we can write a generating function to generate all the basis states of all the IRs of $SU(3)$. This generating function is computationally a very convenient realization of the basis of the model space of $SU(3)$. Moreover, we can define a scalar product on this space by choosing one of the variables, say z_3, to be a planar rotor $\exp(\mathrm{i}\theta)$. Thus the model space for $SU(3)$ is now a Hilbert space with this (auxiliary) scalar product between the basis states. The above construction was carried out in detail in a previous paper by us [1]. For easy accessibility we give a self-contained summary of those results here.

2.4. Labels for the basis states

(i) Gelfand–Zetlein labels. Normalized basis vectors are denoted by $|M, N; P, Q, R, S, U, V\rangle$. All labels are non-negative integers. All irreducible representations (IRs) are uniquely labelled by (M, N). For a given IR (M, N), labels (P, Q, R, S, U, V) take all non-negative integral values subject to the constraints

$$R + U = M \qquad S + V = N \qquad P + Q = R + S\,. \tag{8}$$

The allowed values can be prescribed easily: R takes all values from 0 to M, and S from 0 to N. For a given R and S, Q takes on all values from 0 to $R + S$.

(ii) Quark model labels. The relation between the above Gelfand–Zetlein labels and the quark model labels is as given below,

$$\begin{aligned} &2I = P + Q = R + S \qquad 2I_3 = P - Q \\ &Y = \tfrac{1}{3}(M - N) + V - U = \tfrac{2}{3}(N - M) - (S - R) \end{aligned} \tag{9}$$

where R takes all values from 0 to M. S takes all values from 0 to N. For a given R and S, Q takes all values from 0 to $R + S$.

2.5. Explicit realization of the basis states

(i) Generating function for the basis states of SU(3). The generating function for the basis states of the IR's of $SU(3)$ can be written as

$$g(p, q, r, s, u, v) = \exp(r(pz_1 + qz_2) + s(pw_2 - qw_1) + uz_3 + vw_3)\,. \tag{10}$$

The coefficient of the monomial $p^P q^Q r^R s^S u^U v^V$ in the Taylor expansion of (10), after eliminating w_3 using (7), in terms of these monomials gives the basis state of $SU(3)$ labelled by the quantum numbers P, Q, R, S, U, V.

(ii) Formal generating function for the basis states of SU(3). The generating function (10) can be written formally as

$$g = \sum_{P,Q,R,S,U,V} p^P q^Q r^R s^S u^U v^V |PQRSUV) \tag{11}$$

where $|PQRSTUV)$ is an unnormalized basis state of $SU(3)$ labelled by the quantum numbers P, Q, R, S, U, V.

Note that the constraint $P + Q = R + S$ is automatically satisfied in the formal as well as explicit Taylor expansion of the generating function.

(iii) Generalized generating function for the basis states of SU(3). It is useful, while computing the normalizations (see below) of the basis states, to write the above generating function in the following form:

$$\mathcal{G}(p, q, r, s, u, v) = \exp(r_p z_1 + r_q z_2 + s_p w_2 + s_q w_1 + u z_3 + v w_3)\,. \tag{12}$$

In the above generalized generating function (12) the following notation holds.

$$r_p = rp \qquad r_q = rq \qquad s_p = sp \qquad s_q = -sq\,. \tag{13}$$

2.6. Notation

Hereafter, for simplicity in notation we assume all variables, other than the z_j^i and w_j^i where $i, j = 1, 2, 3$, to be real even though we have treated them as complex variables at some places. Our results are valid even without this restriction as we are interested only in the coefficients of the monomials in these real variables rather than in the monomials themselves.

2.7. Auxiliary scalar product for the basis states.

The scalar product to be defined in this section is auxiliary in the sense that it does not give us the 'true' normalizations of the basis states of $SU(3)$. However, it is computationally very convenient for us as all computations with this scalar product get reduced to simple Gaussian integrations and the 'true' normalizations themselves can then be found quite easily [1].

(i) Scalar product between generating functions of basis states of SU(3). We define the scalar product between any two basis states in terms of the scalar product between the corresponding generating functions as follows:

$$\begin{aligned}(g', g) = &\int_{-\pi}^{+\pi} \frac{\mathrm{d}\theta}{2\pi} \int \frac{\mathrm{d}^2 z_1}{\pi^2} \frac{\mathrm{d}^2 z_2}{\pi^2} \frac{\mathrm{d}^2 w_1}{\pi^2} \frac{\mathrm{d}^2 w_2}{\pi^2} \exp(-\bar{z}_1 z_1 - \bar{z}_2 z_2 - \bar{w}_1 w_1 - \bar{w}_2 w_2) \\ &\times \exp\left((r'(p' z_1 + q' z_2) + s'(p' w_2 - q' w_1) - \frac{-v'}{z_3}(z_1 w_1 + z_2 w_2) + u' \bar{z}_3 \right) \\ &\times \exp\left((r(p z_1 + q z_2) + s(p w_2 - q w_1) - \frac{-v}{z_3}(z_1 w_1 + z_2 w_2) + u z_3 \right) \\ = &(1 - v'v)^{-2} \left(\sum_{n=0}^{\infty} \frac{(u'u)^n}{(n!)^2} \right) \exp\left[(1 - v'v)^{-1}(p'p + q'q)(r'r + s's)\right].\end{aligned} \tag{14}$$

(ii) Choice of the variable z_3. To obtain (14) we have made the choice

$$z_3 = \exp(\mathrm{i}\theta)\,. \tag{15}$$

The choice, equation (15), makes our basis states for $SU(3)$ depend on the variables z_1, z_2, w_1, w_2 and θ.

The just described scalar product can be translated into the language of boson operators and the same results can be obtained [11].

(iii) Scalar product between the generalized generating functions of the basis states of SU(3). For the generalized generating function the scalar product becomes

$$(\mathcal{G}',\mathcal{G}) = (1-v'v)^{-2}\exp\left[(1-v'v)^{-1}(r'_p r_p + r'_q r_q + s'_p s_p + s'_q s_q)\right] \times \left[\sum_{n=0}^{\infty}\frac{1}{(n!)^2}\left(u' - v\frac{(r'_p s'_q + r'_q s'_p)}{(1-v'v)}\right)^n \left(u - v'\frac{(r_p s_q + r_q s_p)}{(1-v'v)}\right)^n\right] \tag{16}$$

and as in (13)

$$\begin{aligned} r_p &= rp \qquad r_q = rq \qquad s_p = sp \qquad s_q = -sq \\ r'_p &= r'p' \qquad r'_q = r'q' \qquad s'_p = s'p' \qquad s'_q = -s'q' \,. \end{aligned} \tag{17}$$

2.8. Normalizations

(i) Auxiliary normalizations of unnormalized basis states. The scalar product between two unnormalized basis states, computed using our auxiliary scalar product, is given by

$$\begin{aligned} M(PQRSUV) &\equiv (PQRSUV|PQRSUV) \\ &= \frac{(V+P+Q+1)!}{P!Q!R!S!U!V!(P+Q+1)}\,. \end{aligned} \tag{18}$$

(ii) Scalar product between the unnormalized and normalized basis states. The scalar product, computed using our auxiliary scalar product, between an unnormalized basis state and a normalized one is given by the next equation where it is denoted by $(PQRSUV||PQRSUV\rangle$:

$$(PQRSUV||PQRSUV\rangle = N^{-1/2}(PQRSUV) \times M(PQRSUV)\,. \tag{19}$$

(iii) 'True' normalizations of the basis states. We call the ratio of the auxiliary norm of the unnormalized basis state represented by $|PQRSUV)$ and the scalar product of the unnormalized basis state with a normalized Gelfand–Zeitlin state, represented by $|PQRSUV\rangle$, 'true' normalization. It is given by

$$\begin{aligned} N^{1/2}(PQRSUV) &\equiv \frac{(PQRSUV|PQRSUV)}{\langle PQRSUV|PQRSUV\rangle} \\ &= \left(\frac{(U+P+Q+1)!(V+P+Q+1)!}{P!Q!R!S!U!V!(P+Q+1)}\right)^{1/2}\,. \end{aligned} \tag{20}$$

3. Generating function for the Wigner D-matrix elements of $SU(3)$

Let us start with (11),

$$g(p,q,r,s,u,v,z_1,z_2,w_1,w_2) = \sum_{P,Q,R,S,U,V} p^P q^Q r^R s^S u^U v^V |PQRSUV) \tag{21}$$

where $|PQRSUV)$ is an unnormalized basis state in the IR labelled by the two positive integers $(M = R+U, N = S+V)$.

We know from (20),

$$|PQRSUV) = N^{(1/2)}(PQRSUV)|PQRSUV\rangle \tag{22}$$

where $2I = P + Q$ and $|PQRSUV)$ is a normalized basis state.

Therefore

$$g = \sum_{PQRSUV} \left(\frac{(U+2I+1)!(V+2I+1)!}{P!Q!R!S!U!V!(2I+1)} \right)^{1/2} p^P q^Q r^R s^S u^U v^V |PQRSUV\rangle . \tag{23}$$

Now, if $A \in SU(3)$, then by definition,

$$\begin{aligned} Ag(p,q,\ldots) = & \sum_{PQRSUV} \sum_{P'Q'R'S'U'V'} \left(\frac{(U+2I+1)!(V+2I+1)!}{P!Q!R!S!U!V!(2I+1)} \right)^{1/2} \\ & \times D^{(M=R+U,\, N=S+V)}_{PQRSUV,\, P'Q'R'S'U'V'}(A) \times p^P q^Q r^R s^S u^U v^V \times |PQRSUV\rangle \end{aligned} \tag{24}$$

where the object $D^{(M=R+U,\, N=S+V)}_{PQRSUV,\, P'Q'R'S'U'V'}(A)$ is the Wigner D-matrix element corresponding to $A \in SU(3)$ in the IR labelled by the integers M, N. Here the quantum numbers $PQRSUV$ and $P'Q'R'S'U'V'$ label the row and column, respectively, of the matrix.

To get a generating function for the matrix elements alone we have to take the inner product of this transformed generating function with the generating function for the basis states. Throughout the following we take the variables p, q, r, s, u, v together with their primed and unprimed variants to be real since we are interested only in the coefficients of monomials in these different sets of variables in different expansions and are not interested in these variables or their functions as such.

Thus,

$$\begin{aligned} & \big(g(p'', q'', r'', s'', u'', v'';\; z_1, z_2, z_3, w_1, w_2),\;\; Ag(p, q, r, s, u, v;\; z_1, z_2, z_3, w_1, w_2)\big) \\ & \quad = \sum_{PQRSUV} \sum_{P'Q'R'S'U'V'} \sum_{P''Q''R''S''U''V''} \left(\frac{(U+2I+1)!(V+2I+1)!}{P!Q!R!S!U!V!(2I+1)} \right)^{1/2} \\ & \qquad \times (P''Q''R''S''U''V'' \| P'Q'R'S'U'V'\rangle \times D^{(M=R+U, N=S+V)}_{PQRSUV, P'Q'R'S'U'V'}(A) \\ & \qquad \times p^P q^Q r^R s^S u^U v^V p''^{P''} q''^{Q''} r''^{R''} s''^{S''} u''^{U''} v''^{V''} . \end{aligned} \tag{25}$$

But we know from (19),

$$\begin{aligned} & (P''Q''R''S''U''V'' || P'Q'R'S'U'V'\rangle \\ & \quad = \left(\frac{(U'+2I'+1)!(V'+2I'+1)!}{P'!Q'!R'!S'!U'!V'!(2I'+1)} \right)^{-1/2} \frac{(V'+P'+Q'+1)!}{P'!Q'!R'!S'!U'!V'!(P'+Q')} \\ & \qquad \times \delta_{P''P'} \delta_{Q''Q'} \delta_{R''R'} \delta_{S''S'} \delta_{U''U'} \delta_{V''V'} . \end{aligned} \tag{26}$$

Substituting this formula and changing the double primed variables to single primed ones, we get

$$\begin{aligned} & \big(g(p', q', r,' s', u', v';\; z_1, z_2, z_3, w_1, w_2),\; Ag(p, q, r, s, u, v;\; z_1, z_2, z_3, w_1, w_2)\big) \\ & \quad = \sum_{PQRSUV;\, P'Q'R'S'U'V'} \left(\frac{(U+2I+1)!(V+2I+1)!}{P!Q!R!S!U!V!(2I+1)} \right) \\ & \qquad \times \left(\frac{P'!Q'!R'!S'!U'!V'!(2I'+1)}{(U'+2I'+1)!(V'+2I'+1)!} \right)^{1/2} \\ & \qquad \times \left(\frac{(V'+P'+Q'+1)!}{P'!Q'!R'!S'!U'!V'!(P'+Q'+1)} \right) \times D^{(M=R+U, N=S+V)}_{PQRSUV, P'Q'R'S'U'V'}(A) \\ & \qquad \times p^P q^Q r^R s^S u^U v^V p'^{P'} q'^{Q'} r'^{R'} s'^{S'} u'^{U'} v'^{V'} . \end{aligned} \tag{27}$$

We therefore conclude that the Wigner D-matrix element,

$$D^{(M=R+U,\, N=S+V)}_{PQRSUV, P'Q'R'S'U'V'}$$

for $SU(3)$ can be obtained as the coefficient of the monomial,

$$p^P q^Q r^R s^S u^U v^V \times p'^{P'} q'^{Q'} r'^{R'} s'^{S'} u'^{U'} v'^{V'}$$

multiplied by

$$\left(\frac{P!Q!R!S!U!V!(2I+1)}{(U+2I+1)!(V+2I+1)!} \times \frac{(U'+2I'+1)!(V'+2I'+1)!}{P'!Q'!R'!S'!U'!V'!(2I'+1)}\right)^{1/2} \times \left(\frac{P'!Q'!R'!S'!U'!V'!(P'+Q'+1)}{(V'+P'+Q'+1)!}\right) \tag{28}$$

in the inner product (g', Ag) between the untransformed and transformed generating functions for the basis states.

Next we calculate this inner product using the explicit realization for the generating function. For this purpose it is advantageous, as will be seen soon, to use the generalized generating function for the basis states

$$\begin{aligned}\mathcal{G} &= \exp(r_p z_1 + r_q z_2 + s_p w_2 + s_q w_1 + u z_3 + v w_3) \\ &= \exp\left((\, r_p r_q u\,) \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} + (\, w_1 w_2 w_3\,) \begin{pmatrix} s_q \\ s_p \\ v \end{pmatrix} \right).\end{aligned} \tag{29}$$

When any element $A \in SU(3)$ acts on this generating function it undergoes the following transformation:

$$A\mathcal{G} = \exp\left((\, r_p \quad r_q \quad u\,) A \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} + (\, w_1 \quad w_2 \quad w_3\,) A^{\dagger} \begin{pmatrix} s_q \\ s_p \\ v \end{pmatrix} \right). \tag{30}$$

As is clear from the above equation we can let the triplets r_p, r_q, u and s_q, s_p, v undergo the transformation instead of the triplets z_1, z_2, z_3 and w_1, w_2, w_3. Therefore we can write the transformed generating function as

$$A\mathcal{G} = \mathcal{G}(r_p'', r_q'', u''; s_q'', s_p'', v'') \tag{31}$$

where

$$\begin{aligned} r_p'' &= a_{11} r_p + a_{21} r_q + a_{31} u \\ r_q'' &= a_{12} r_p + a_{22} r_q + a_{32} u \\ u'' &= a_{13} r_p + a_{23} r_q + a_{33} u \\ s_q'' &= a_{11}^* s_q + a_{21}^* s_p + a_{31}^* v \\ s_p'' &= a_{12}^* s_q + a_{22}^* s_p + a_{32}^* v \\ v'' &= a_{13}^* s_q + a_{23}^* s_p + a_{33}^* v\,. \end{aligned} \tag{32}$$

To continue with our computation we have to take the inner product of this transformed generating function with the (untransformed) generating function of the basis states.

This is known to us from (16) as

$$(\mathcal{G}', \mathcal{G}'') = (1 - v'v'')^{-2} \exp\left[(1 - v'v'')^{-1}(r_p' r_p'' + r_q' r_q'' + s_p' s_p'' + s_q' s_q'')\right] \times \left[\sum_{n=0}^{\infty} \frac{1}{(n!)^2} \left(u' - v'' \frac{(r_p' s_q' + r_q' s_p')}{(1 - v'v'')} \right)^n \left(u'' - v' \frac{(r_p'' s_q'' + r_q'' s_p'')}{(1 - v'v'')} \right)^n \right]. \tag{33}$$

This expression gets further simplified if we substitute from (13)

$$r_p' = r'p' \qquad r_q' = r'q' \qquad s_q' = -s'q' \qquad s_p' = s'p\,.$$

We therefore get

$$(\mathcal{G}', \mathcal{G}'') = (1 - v'v'')^{-2} \exp\left[(1 - v'v'')^{-1}(r'_p r''_p + r'_q r''_q + s'_p s''_p + s'_q s''_q)\right] \times \left[\sum_{n=0}^{\infty} \frac{1}{(n!)^2}(u')^n (u'' - v' \frac{(r''_p s''_q + r''_q s''_p)}{(1 - v'v'')})^n\right]. \tag{34}$$

One last simplification can be brought about in the above expression when we recognize that

$$\begin{aligned} r''_p s''_q + r''_q s''_p + u''v'' &= r_p s_q + r_q s_p + vu \\ &= vu . \end{aligned} \tag{35}$$

This tells us that

$$r''_p s''_q + r''_q s''_p = uv - u''v'' . \tag{36}$$

Substituting this in our expression (34) for the inner product we get

$$\begin{aligned} (\mathcal{G}', \mathcal{G}'') &= (1 - v'v'')^{-2} \exp\left[(1 - v'v'')^{-1}(r'_p r''_p + r'_q r''_q + s'_p s''_p + s'_q s''_q)\right] \\ &\quad \times \left[\sum_{n=0}^{\infty} \frac{1}{(n!)^2}(u')^n (u'' - v' \frac{(uv - u''v'')}{(1 - v'v'')})^n\right] \\ &= (1 - v'v'')^{-2} \exp\left[(1 - v'v'')^{-1}(r'_p r''_p + r'_q r''_q + s'_p s''_p + s'_q s''_q)\right] \\ &\quad \times \left[\sum_{n=0}^{\infty} \frac{1}{(n!)^2}\left(u' \frac{(u'' - uvv')}{(1 - v'v'')}\right)^n\right]. \end{aligned} \tag{37}$$

On the other hand if we use a slightly modified scalar product [12] for the θ part (only) and use the symbol $\mathcal{G}(\mathcal{D}(\mathcal{A}))$ for the generating function, where $D(A)$ is the Wigner D-matrix for $SU(3)$ in an arbitrary representation, then

$$\begin{aligned} \mathcal{G}(\mathcal{D}(\mathcal{A})) &= (1 - v'v'')^{-2} \exp\left[\frac{(r'_p r''_p + r'_q r''_q + s'_p s''_p + s'_q s''_q)}{(1 - v'v'')}\right. \\ &\quad \left. + \left(u' - v'' \frac{(r'_p s'_q + r'_q s'_p)}{(1 - v'v'')}\right)\left(u'' - v' \frac{(r''_p s''_q + r''_q s''_p)}{(1 - v'v'')}\right)\right] \\ &= (1 - v'v'')^{-2} \exp\left[\frac{(r'_p r''_p + r'_q r''_q + s'_p s''_p + s'_q s''_q) + u'(u'' - uvv')}{(1 - v'v'')}\right]. \end{aligned} \tag{38}$$

The expression on the right-hand side of (37) or of (38) is our generating function for the Wigner D-matrix elements of $SU(3)$.

3.1. Symmetries of the D-matrix elements

The form of the generating function $\mathcal{G}(\mathcal{D}(\mathcal{A}))$ given in (38) is a convenient starting point for the discussion of the symmetries of the D-matrix [10]. The symmetries of the Wigner D-matrix play a very important role in the study of the special functions connected with $SU(3)$ as the D-matrix elements themselves happen to be precisely the special functions of $SU(3)$. One usual way to obtain the special functions is as solutions of the appropriate differential equations. But here we have obtained them by combining their tensorial properties with the technique of using a generating function. It is needless to say that these symmetries form a crucial part of the topic of harmonic analysis on $SU(3)$ also. We will pursue these subjects elsewhere.

4. Wigner's D-matrix elements of $SU(3)$ in any irreducible representation

In this section our task is to extract the coefficient of the monomial

$$p^P q^Q r^R s^S u^U v^V \times p'^{P'} q'^{Q'} r'^{R'} s'^{S'} u'^{U'} v'^{V'}$$

in the expansion of the generating function that we have obtained above, equation (37), for the Wigner D-matrix elements of $SU(3)$. For this purpose we expand the right-hand side of the above generating function and obtain

$$\begin{aligned}
&\sum_{m=0}^{\infty} \frac{(r'_p r''_p + r'_q r''_q + s'_p s''_p + s'_q s''_q)}{m!(1-v'v'')^m} \sum_{n=0}^{\infty} \frac{1}{(n!)^2} \left(u' \frac{(u''-uvv')}{(1-v'v'')^{(1+2/n)}} \right)^n \\
&\quad = \sum_{m=n=s=0}^{\infty} \sum_{t,m_1,m_2,m_3=0}^{n,m,m-m_1,m-m_1-m_2} \\
&\quad \times \frac{(s+m+n+1)!}{m!n!(n-t)!t!(m+n+1)!s!(m-m_1-m_2-m_3)!m_3!} \\
&\quad \times (r''_p)^{m_1} (r''_q)^{m_2} (s''_p)^{m_3} (s''_q)^{m-\sum m_i} (p')^{m_1+m_3} (q')^{m-m_1-m_3} (r')^{m_1+m_2} (s')^{m-m_1-m_2} \\
&\quad \times (u')^n (v')^{n-t+s} u''^n v'^{n-t+s} u''^t v''^s (-uv)^{n-t} .
\end{aligned} \tag{39}$$

Now let,

$$\begin{aligned}
&m_1 + m_2 = P' \qquad m - m_1 - m_3 = Q' \qquad m_1 + m_2 = R' \\
&m - m_1 - m_2 = S' \qquad n = U' \qquad n - t + s = V' .
\end{aligned} \tag{40}$$

The above assignments imply

$$\begin{aligned}
&m = P' + Q' \qquad m_2 - m_3 = R' - P' \\
&m_2 = R' - P' + m_3 \qquad s = t + V' - U' .
\end{aligned} \tag{41}$$

This gives us

$$\begin{aligned}
D^{(M=R+U,\, N=S+V)}_{PQRSUV,\, P'Q'R'S'U'V'}(A) &= \sum_{P'+Q'=0}^{\infty} \sum_{U'=0}^{\infty} \sum_{V'=U'}^{\infty} \sum_{m_1=0}^{P'+Q} \sum_{t=0}^{U'} \sum_{m_3=0}^{S'} \sum_{m_2=0}^{P'+Q'-m_1} \\
&\times \frac{(t+V'-U')!(-uv)^{U'-t}}{(P'+Q')!U'!(U'-t)!t!(P'+Q'+U'+1)!} \\
&\times \frac{1}{(t+V'-U')!(R'-P'+m_3)!(S'-m_3)!m_3!} \\
&\times (r''_p)^{m_1} (r''_q)^{m_2} (s''_p)^{m_3} (s''_q)^{S'-m_3} u''^t v''^{t+V'-U'} \\
&\times (p')^{P'} (q')^{Q'} (r')^{R'} (s')^{S'} (u')^{U'} (v')^{V'} .
\end{aligned} \tag{42}$$

In the above we substitute for the following from (32):

$$r''_p, \quad r''_q, \quad u'', \quad s''_q, \quad s''_p, \quad v''$$

and get

$$\begin{aligned}
D^{(M=R+U,\, N=S+V)}_{PQRSUV,\, P'Q'R'S'U'V'}(A) &= \sum_{P'+Q'=0}^{\infty} \sum_{U'=0}^{\infty} \sum_{V'=U'}^{\infty} \sum_{m_1=0}^{P'+Q} \sum_{t=0}^{U'} \sum_{m_3=0}^{S'} \sum_{m_2=0}^{P'+Q'-m_1} \\
&\times \sum_{m_{11}+m_{12}+m_{13}=m_1} \sum_{m_{21}+m_{22}+m_{23}=m_2} \sum_{m_{31}+m_{32}+m_{33}=m_3} \\
&\times \sum_{m_{41}+m_{42}+m_{43}=S'-m_3} \sum_{t_{11}+m_{12}+m_{13}=t} \sum_{t_{21}+t_{22}+t_{23}=t+V'-U'}
\end{aligned}$$

$$\times\frac{(-1)^{U'-t}(t+V'-U')!(-uv)^{U'-t}}{(P'+Q')!U'!(U'-t)!t!(P'+Q'+U'+1)!}$$
$$\times\frac{1}{(t+V'-U')!(R'-P'+m_3)!(S'-m_3)!m_3!}$$
$$\times\frac{m_1!m_2!m_3!(S'-m_3)!t!(t+V'-U')!}{m_{11}!m_{12}!m_{13}!m_{21}!m_{22}!m_{23}!m_{31}!m_{32}!m_{33}!m_{41}!m_{42}!m_{43}!t_{11}!t_{12}!t_{13}!t_{21}!t_{22}!t_{23}!}$$
$$\times(a_{11})^{m_{11}}(a_{11}^*)^{m_{11}}(a_{21})^{m_{12}}(a_{21}^*)^{m_{42}}(a_{31})^{m_{13}}(a_{31}^*)^{m_{43}}(a_{12})^{m_{21}}(a_{12}^*)^{m_{31}}(a_{22})^{m_{22}}$$
$$\times(a_{22}^*)^{m_{32}}(a_{32})^{m_{23}}(a_{32}^*)^{m_{33}}(a_{13})^{t_{11}}(a_{13}^*)^{t_{21}}(a_{23})^{t_{12}}(a_{23}^*)^{t_{22}}(a_{33})^{t_{13}}(a_{33}^*)^{t_{23}}$$
$$\times(p')^{P'}(q')^{Q'}(r')^{R'}(s')^{S'}(u')^{U'}(v')^{V'}\times(p)^{P}(q)^{Q}(r)^{R}(s)^{S}(u)^{U}(v)^{V} \qquad (43)$$

where we have made the identifications

$$\begin{aligned}
m_{11}+m_{21}+t_{11}+m_{32}+m_{42}+t_{22}&=P\\
m_{12}+m_{22}+t_{12}+m_{41}+m_{32}+t_{22}&=Q\\
m_{11}+m_{21}+t_{11}+m_{12}+m_{22}+t_{22}&=R\\
m_{41}+m_{31}+t_{21}+m_{42}+m_{32}+t_{22}&=S\\
m_{13}+m_{23}+t_{13}+U'-t&=U\\
m_{43}+m_{33}+t_{23}+U'-t&=V\,.
\end{aligned} \qquad (44)$$

Finally, we get the desired object, i.e. the Wigner D-matrix or the finite transformation matrix of the group $SU(3)$ in any irreducible representation by multiplying the above matrix element by the factor in (28).

So finally,

$$D^{(M=R+U,\,N=S+V)}_{PQRSUV,\,P'Q'R'S'U'V'}(A)=\left(\frac{P!Q!R!S!U!V!(2I+1)}{(U+2I+1)!(V+2I+1)!}\,\frac{(U'+2I'+1)!(V'+2I'+1)!}{P'!Q'!R'!S'!U'!V'!(2I'+1)}\right)^{1/2}$$
$$\times\left(\frac{P'!Q'!R'!S'!U'!V'!(P'+Q'+1)}{(V'+P'+Q'+1)!}\right)$$
$$\times\sum_{P'+Q'=0}^{\infty}\sum_{U'=0}^{\infty}\sum_{V'=U'}^{\infty}\sum_{m_1=0}^{P'+Q}\sum_{t=0}^{U'}\sum_{m_3=0}^{S'}\sum_{m_2=0}^{P'+Q'-m_1}\sum_{m_{11}+m_{12}+m_{13}=m_1}\sum_{m_{21}+m_{22}+m_{23}=m_2}$$
$$\times\sum_{m_{31}+m_{32}+m_{33}=m_3}\sum_{m_{41}+m_{42}+m_{43}=S'-m_3}\sum_{t_{11}+m_{12}+m_{13}=t}\sum_{t_{21}+t_{22}+t_{23}=t+V'-U'}$$
$$\times\frac{(-1)^{U'-t}(t+V'-U')!(-uv)^{U'-t}}{(P'+Q')!U'!(U'-t)!t!(P'+Q'+U'+1)!}$$
$$\times\frac{1}{(t+V'-U')!(R'-P'+m_3)!(S'-m_3)!m_3!}$$
$$\times\frac{m_1!m_2!m_3!(S'-m_3)!t!(t+V'-U')!}{m_{11}!m_{12}!m_{13}!m_{21}!m_{22}!m_{23}!m_{31}!m_{32}!m_{33}!m_{41}!m_{42}!m_{43}!}$$
$$\times\frac{1}{t_{11}!t_{12}!t_{13}!t_{21}!t_{22}!t_{23}!}$$
$$\times(a_{11})^{m_{11}}(a_{11}^*)^{m_{11}}(a_{21})^{m_{12}}(a_{21}^*)^{m_{42}}(a_{31})^{m_{13}}(a_{31}^*)^{m_{43}}(a_{12})^{m_{21}}(a_{12}^*)^{m_{31}}(a_{22})^{m_{22}}$$
$$\times(a_{22}^*)^{m_{32}}(a_{32})^{m_{23}}(a_{32}^*)^{m_{33}}(a_{13})^{t_{11}}(a_{13}^*)^{t_{21}}(a_{23})^{t_{12}}(a_{23}^*)^{t_{22}}\ (a_{33})^{t_{13}}(a_{33}^*)^{t_{23}}\Big)\,. \qquad (45)$$

The above equation, equation (45) for the Wigner D-matrix element for $SU(3)$ is the analogue of Wigner's D-matrix element for $SU(2)$ (see, for example, [9, 10]). One can try to reduce the number of summations in the above formula for the D-matrix elements but we will not attempt it here.

4.1. Parametrization using $\vec{Z}$ and $\vec{W}$

Our expression, equation (45), is independent of any particular parametrization of the elements of the matrix $A \in SU(3)$. We now give a parametrization [3] of $A \in SU(3)$ in terms of the complex variables $z_, z_2, z_3$ and w_1, w_2, w_3 corresponding to the IRs $\underline{3}$ and $\underline{3}^*$.

For this purpose we constrain these variables to the intersection of the two unit 5-spheres

$$\begin{aligned} &|z_1|^2 + |z_2|^2 + |z_3|^2 = 1 \\ &|w_1|^2 + |w_2|^2 + |w_3|^2 = 1 \end{aligned} \tag{46}$$

the complex unit cone

$$z_1 w_1 + z_2 w_2 + z_3 w_3 = 0\,. \tag{47}$$

Then $A \in SU(3)$ can be written as below

$$A = \begin{pmatrix} z_1^* & z_2^* & z_3^* \\ w_1 & w_2 & w_3 \\ u_1 & u_2 & u_3 \end{pmatrix} \tag{48}$$

where

$$u_i = \sum_{j,k} \epsilon_{ijk} z_j w_k^*\,. \tag{49}$$

The following two points are to be noted. (i) This unit cone is a homogeneous space [3] for the action of the group $SU(3)$ and (ii) the group manifold itself can be identified with this cone. This is contrary to the popular belief that only in the case of $SU(2)$ can the group manifold be identified with a geometric surface. Moreover, as can be seen from the section on the review of our earlier results this cone serves as a model space for the IRs of $SU(3)$.

5. Discussion

In this paper, making use of the tools of a calculus that we had set up previously to do computations on $SU(3)$, we have obtained (i) a generating function (equations (37) and (38)) for the Wigner D-matrix elements of $SU(3)$ and (ii) a closed-form algebraic expression (equation (45)) for the individual Wigner D-matrix elements of $SU(3)$ in any irreducible representation. To our knowledge this is the first time that such a generating function has been written for $SU(3)$. See also Hage Hassan [7] and references therein. But this generating function gives us unitary matrix elements of $SU(3)$ only up to a multiplicative factor. The reason for this is that our auxiliary measure for the basis states is not a group-invariant measure. This is clearly a drawback. However, for computing objects such as the group characters this is no hurdle since the characters are invariant under basis transformations. In fact, one can write a generating function for the characters of $SU(3)$ and from that one can derive the Weyl's character formula [12]. Since it is possible to generalize our results (see the overview) to higher groups it should also be possible to obtain the Wigner D-matrix elements for the higher groups using methods similar to the one described in this paper.

We also note that our generating function is in fact a product of two factors, one of which is an exponential function and the second is a power series. This seems to be a consequence of the particular choice of variables occurring in the construction of our basis functions and also because of the particular scalar product that was used by us. As a result the θ variable part of the formula for the D-matrix elements decouples from the part that depends on other variables. By using a slightly different scalar product for the θ part of the basis functions one can obtain the generating function for the D-matrix elements as a single exponential function alone (see equation (38)). But there is no guarantee that such a scalar product will also yield an equally elegant formulae for other objects such as, for example, for Clebsch–Gordan coefficients for which the first scalar product itself yields a single exponential as a generating function [1]. Next, the expression for the individual D-matrix elements for $SU(3)$ has been obtained by many people previously [4–6]. But one desirable feature about our expression is that it is quite compact and is independent of any particular parametrization used for describing the defining representation of $SU(3)$.

Now, a word regarding the generating function technique and its possible usage to compute some physical quantities. Stated briefly, the technique involves writing a generating function for all objects of interest (such as all the D-matrix elements in all IRs) and then extracting the object of interest for the current physical problem (such as the D-matrix element in a particular IR) as a coefficient of the appropriate monomial in the power-series expansion of the generating function in terms of these monomials. Therefore it may be useful to try to apply this technique to write a generating function for a partition function involving $SU(3)$ IRs, for example, for partition functions occurring in QGP [13]. Then one can try to extract that part of the partition function which corresponds to the physically interesting case such as, for example, the partition function corresponding to the $SU(3)$ scalar representation etc, by the above-mentioned technique. The usual way of extracting such terms is by making use of the orthogonality properties of characters of $SU(3)$ but the resultant class integrals cannot be solved exactly so one has to resort to some approximations. The generating function method can obviate the necessity of such steps and the resultant approximations. This application is presently under investigation and will be the subject of a future publication.

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Appendix A. Examples

To compute the matrix elements of $SU(3)$, for lower dimensions, it is easier to work with the generating function for the matrix elements (equations (37) and (38)).

For the irreducible representation $\underline{3}$ the only terms of the generating function which are relevant are the ones linear in the primed and and doubly primed composite variables $r'_p, r''_p, \ldots$. This gives us the following expansion for the generating function:

$$r'_p r''_p + r'_q r''_q + s'_p s''_p + s'_q s''_q + u'u'' + v'v'' \,. \tag{A1}$$

We now substitute for the doubly primed variables, in the above expression, from equations (32) and (17), and extract the coefficients of the various monomials $p^P q^Q r^R s^S u^U v^V$ for the values of the quantum numbers P, Q, R, S, U, V given in the table below for the IR $\underline{3}$. This gives us the $SU(3)$ representative matrix (equation (1)).

Table A1. $\underset{\sim}{3}(M=1, N=0)$.

	P	Q	R	S	U	V	I	I_3	Y	$\|PQRSTUV)$	$N^{1/2}$
u	1	0	1	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{3}$	z_1	$\sqrt{2}$
d	0	1	1	0	0	0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{3}$	z_2	$\sqrt{2}$
s	0	0	0	0	1	0	0	0	$-\frac{2}{3}$	z_3	$\sqrt{2}$

Table A2. $\underset{\sim}{3}(M=1, N=0)$.

	u	d	s
u	a_{11}	a_{12}	a_{13}
d	a_{21}	a_{22}	a_{23}
s	a_{31}	a_{32}	a_{33}

A similar treatment for the IR $\underset{\sim}{3}^*$, using the corresponding table, given below, gives us the $SU(3)$ matrix $A^\dagger$.

Table A3. $\underset{\sim}{3}^*(M=0, N=1)$.

	P	Q	R	S	U	V	I	I_3	Y	$\|PQRSTUV)$	$N^{1/2}$
$\bar{d}$	1	0	0	1	0	0	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{3}$	w_2	$\sqrt{2}$
$\bar{u}$	0	1	0	0	0	0	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{3}$	$-w_1$	$\sqrt{2}$
$\bar{s}$	0	0	0	0	0	1	0	0	$\frac{2}{3}$	w_3	$\sqrt{2}$

Table A4. $\underset{\sim}{3}^*(M=0, N=1)$.

	$\bar{d}$	$\bar{u}$	$\bar{s}$
$\bar{d}$	a^*_{11}	a^*_{21}	a^*_{31}
$\bar{u}$	a^*_{12}	a^*_{22}	a^*_{32}
$\bar{s}$	a^*_{13}	a^*_{23}	a^*_{33}

We now treat the case of the IR $\underset{\sim}{8}$. The terms relevant for this IR are quadratic in the primed and doubly primed composite variables. For example the first term in the expansion of the generating function (38) is

$$r'_p r''_p s'_p s''_p = p'^2 r' s' \big(-A_{11}A^*_{11}pqrs + A_{11}A^*_{12}A^*_{12}p^2rs + A_{11}A^*_{13}prv - A_{21}A^*_{12}q^2rs \\ + A_{21}A^*_{12}pqrs + A_{21}A^*_{13}qrv + A_{31}A^*_{11}qsu + A_{31}A^*_{12}psu + A_{31}A^*_{13}uv \big) \,. \tag{A2}$$

In equation (A2) the various monomials $p^P q^Q r^R s^S u^U v^V$ correspond to the quantum numbers P, Q, R, S, U, V in the first row of the table corresponding to the IR $\underset{\sim}{8}$ as indicated in the table given below. Therefore their coefficients give us the first row of the $SU(3)$ Wigner D-matrix for the IR $\underset{\sim}{8}$. One can build the remaining rows in a similar manner. The result is given in the form of a table below.

Table A5. $\underline{8}(M = 1, N = 1)$.

	P	Q	R	S	U	V	I	I_3	Y	$\vert PQRSTUV\rangle$	$N^{1/2}$
π^+	2	0	1	1	0	0	1	1	0	z_1w_2	$\sqrt{6}$
π^0	1	1	1	1	0	0	1	0	0	$-z_1w_1 + z_2w_2$	$\sqrt{12}$
π^-	0	2	1	1	0	0	1	-1	0	$-z_2w_1$	$\sqrt{6}$
K^+	1	0	1	0	0	1	$\frac{1}{2}$	$\frac{1}{2}$	1	z_1w_3	$\sqrt{6}$
K^0	0	1	1	0	0	1	$\frac{1}{2}$	$-\frac{1}{2}$	1	z_2w_3	$\sqrt{6}$
$\bar{K}^0$	1	0	0	1	1	0	$\frac{1}{2}$	$\frac{1}{2}$	-1	w_2z_3	$\sqrt{6}$
K^-	0	1	0	1	1	0	$\frac{1}{2}$	$-\frac{1}{2}$	-1	$-w_1z_3$	$\sqrt{6}$
η	0	0	0	0	1	1	0	0	0	$(z_3w_3 = -z_1w_1 - z_2w_2)$	2

Table A6. $\underline{8}^*(M = 1, N = 1)$.

	π^+	π^0	π^-	K^+	K^0	$\bar{K}^0$	K^-	η
	π^+	π^0	π^-	K^+	K^0	$\bar{K}^0$	K^-	η
π^+	$(a_{21}a_{12}^* - a_{11}a_{11}^*)$	$\frac{a_{11}a_{12}^*}{\sqrt{2}}$	$a_{11}a_{13}^*$	$\frac{-\sqrt{2}a_{21}a_{12}^*}{3}$	$\frac{\sqrt{2}a_{21}a_{13}^*}{3}$	$-\sqrt{2}a_{31}a_{11}^*$	$\sqrt{2}a_{31}a_{12}^*$	$\frac{a_{31}a_{13}^*}{\sqrt{3}}$
π^0	$\frac{(a_{21}a_{21}^* - a_{11}a_{11}^*)}{\sqrt{2}}$	$\frac{a_{11}a_{21}^*}{2}$	$\frac{a_{11}a_{31}^*}{\sqrt{2}}$	$-\frac{a_{21}a_{11}^*}{6\sqrt{2}}$	$\frac{a_{21}a_{31}^*}{6\sqrt{2}}$	$-\frac{a_{31}a_{11}^*}{\sqrt{2}}$	$\frac{a_{31}a_{21}^*}{\sqrt{2}}$	$\frac{a_{31}a_{31}^*}{2\sqrt{3}}$
π^-	$(a_{22}a_{21}^* - a_{12}a_{11}^*)$	$\frac{a_{12}a_{21}^*}{\sqrt{2}}$	$a_{12}a_{31}^*$	$-\frac{\sqrt{2}a_{22}a_{11}^*}{3}$	$\frac{\sqrt{2}a_{22}a_{31}^*}{3}$	$-\sqrt{2}a_{32}a_{11}^*$	$\sqrt{2}a_{32}a_{21}^*$	$\frac{a_{32}a_{31}^*}{\sqrt{3}}$
K^+	$(a_{21}a_{23}^* - a_{11}a_{13}^*)$	$\frac{a_{11}a_{23}^*}{\sqrt{2}}$	$\frac{a_{11}a_{33}^*}{\sqrt{2}}$	$-\frac{a_{21}a_{13}^*}{3}$	$\frac{a_{21}a_{33}^*}{3}$	$-a_{31}a_{13}^*$	$a_{31}a_{23}^*$	$\frac{a_{31}a_{33}^*}{\sqrt{6}}$
K^0	$(a_{21}a_{23}^* - a_{12}a_{13}^*)$	$\frac{a_{12}a_{23}^*}{\sqrt{2}}$	$a_{12}a_{33}^*$	$-\frac{a_{21}a_{13}^*}{3}$	$\frac{a_{21}a_{33}^*}{3}$	$-a_{31}a_{13}^*$	$a_{31}a_{23}^*$	$\frac{a_{31}a_{33}^*}{\sqrt{6}}$
$\bar{K}^0$	$(a_{23}a_{22}^* - a_{13}a_{12}^*)$	$\frac{a_{13}a_{22}^*}{\sqrt{2}}$	$a_{13}a_{32}^*$	$-\frac{a_{23}a_{12}^*}{3}$	$\frac{a_{23}a_{32}^*}{3}$	$-a_{33}a_{12}^*$	$a_{33}a_{22}^*$	$\frac{a_{33}a_{32}^*}{\sqrt{6}}$
K^-	$(a_{23}a_{21}^* - a_{13}a_{11}^*)$	$\frac{a_{13}a_{21}^*}{\sqrt{2}}$	$a_{13}a_{31}^*$	$-\frac{a_{23}a_{11}^*}{3}$	$\frac{a_{23}a_{31}^*}{3}$	$-a_{33}a_{11}^*$	$a_{33}a_{21}^*$	$\frac{a_{33}a_{31}^*}{\sqrt{6}}$
η	$\frac{\sqrt{3}(a_{23}a_{23}^* - a_{13}a_{13}^*)}{\sqrt{2}}$	$\frac{\sqrt{3}a_{13}a_{23}^*}{2}$	$\frac{\sqrt{3}a_{13}a_{33}^*}{\sqrt{2}}$	$-\frac{a_{23}a_{13}^*}{\sqrt{6}}$	$\frac{a_{23}a_{33}^*}{\sqrt{6}}$	$-\frac{\sqrt{3}a_{33}a_{13}^*}{\sqrt{2}}$	$\frac{\sqrt{3}a_{33}a_{23}^*}{\sqrt{2}}$	$\frac{a_{33}a_{33}^*}{2}$

In all the above computations a normalization factor for each D-matrix element is computed with the help of (28).

References

[1] Prakash J S and Sharatchandra H S 1996 Institute of Mathematical Sciences, Madras, India *Preprint* IMSC 93/26 *J. Math. Phys.* (to appear)

[2] Murnaghan F D 1962 *The Unitary and Rotation Groups* (Washington, DC: Spartan Books)

[3] Beg M A and Ruegg H 1965 *J. Math. Phys.* **6** 677

[4] Chacon E and Moshinsky M 1966 *Phys. Lett.* **23** 567

[5] Nelson T J 1967 *J. Math. Phys.* **8** 857

[6] Holland D F 1969 *J. Math. Phys.* **10** 531

[7] Hage Hassan M 1979 *J. Phys. A: Math. Gen.* **12** 1633; 1983 *J. Phys. A: Math. Gen.* **16** 1835, 2891

[8] Bulgac A and Kusnezov D 1990 *Ann. Phys.* **199** 187

[9] Talman J D 1968 *Special Functions* (New York: Benjamin)

[10] Schwinger J 1952 *On Angular Momentum* US Atomic Energy Commission NYO 3071, unpublished; reprinted 1969 *Quantum Theory of Angular Momentum* ed L C Biedenharn and Van Dam (New York: Academic)

[11] Prakash J S An auxiliary 'differential measure' for $SU(3)$ *Preprint* IP-BBSR-95/76

[12] Prakash J S Weyl's character formula for $SU(3)$—a generating function approach *Preprint* IP-BBSR-96/30, hep-th/9604029

[13] Greiner Muller 1992 *Quantum Mechanics (Symmetries)* 2nd edn (Berlin: Springer)